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Contact fiber bundles

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Abstract

We define contact fiber bundles and investigate conditions for the existence of contact structures on the total space of such a bundle. The results are analogous to minimal coupling in symplectic geometry. The two applications are construction of K -contact manifolds generalizing Yamazaki's fiber join construction and a cross-section theorem for contact moment maps.

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1. Introduction

A few years ago I came across an interesting paper by Yamazaki [7] in which K -contact manifolds were constructed “by fiber join”. The contact manifolds in question were odd-dimensional sphere bundles over Riemann surfaces. These bundles were associated to certain principal torus bundles, and the choice of a contact form on the total space involved a choice of a connection on the torus bundle whose curvature had to satisfy certain nondegeneracy condition. It all very much looked like a contact version of Sternberg's minimal coupling construction in symplectic geometry. The goal of the present paper is to explain why this is indeed the case. In order to do this systematically I felt it is necessary to first sort out the definition of a contact fiber bundle and to investigate conditions for the existence of contact structures on the total space of such a bundle. Of course, since any co-oriented contact manifold is the quotient of a symplectic cone by dilations and since any symplectic cone is a symplectization of a contact manifold, one can argue that contact fiber bundles are

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simply symplectic fiber bundles with fibers being symplectic cones. However, it seems useful to study the matter completely in contact terms. For example, the notion of a K -contact structure does not translate naturally into symplectic terms.

Remark 1.1. In this paper we assume that all contact structures are co-oriented. Recall that a contact structure ξ on a manifold M is *co-oriented* if there exists a 1-form α with $\ker \alpha = \xi$. Equivalently ξ is co-oriented if its annihilator $\xi^\circ \subset T^*M$ is an oriented line bundle. That is, the line bundle minus the zero section, $\xi^\circ \setminus M$, has two components and we have singled out one of the components, call it ξ°_+ . We thus may think of ξ°_+ as a co-orientation of ξ . Note that the image of $\alpha : M \rightarrow TM$ with $\ker \alpha = \xi$ singles out one of the components of $\xi^\circ \setminus M$. The same remark applies to any codimension 1 distribution ξ on M , not just to contact ones. That is, we only consider co-oriented distributions.

A note on notation. If U is a subspace of a vector space V , we denote its annihilator in the dual vector space V^* by U° . Thus $U^\circ = \{\ell \in V^* | \ell|_U = 0\}$. We use the same notation for distributions.

Throughout the paper, the Lie algebra of a Lie group denoted by a capital Roman letter is denoted by the same small letter in the fraktur font: thus \mathfrak{g} denotes the Lie algebra of a Lie group G etc. The vector space dual to \mathfrak{g} is denoted by \mathfrak{g}^* . The identity element of a Lie group is denoted by 1. The natural pairing between \mathfrak{g} and \mathfrak{g}^* is denoted by $\langle \cdot, \cdot \rangle$.

When a Lie group G acts on a manifold M we denote the action by an element $g \in G$ on a point $x \in M$ by $g \cdot x$; $G \cdot x$ denotes the G -orbit of x and so on. The vector field induced on M by an element X of the Lie algebra \mathfrak{g} of G is denoted by X_M (that is, $X_M(x) = (d/dt)|_0(\exp tX) \cdot x$) and the diffeomorphism induced by $g \in G$ on M by g_M . Thus in this notation $g \cdot x = g_M(x)$. The isotropy group of a point $x \in M$ is denoted by G_x ; the Lie algebra of G_x is denoted by \mathfrak{g}_x and is referred to as the isotropy Lie algebra of x . Recall that $\mathfrak{g}_x = \{X \in \mathfrak{g} | X_M(x) = 0\}$.

If X is a vector field and τ is a tensor, then $L_X \tau$ denotes the Lie derivative of τ with respect to X .

If P is a principal G -bundle then $[p, m]$ denotes the point in the associated bundle $P \times_G M = (P \times M)/G$ which is the orbit of $(p, m) \in P \times M$.

2. What is a contact fiber bundle?

Let F be a manifold with a contact distribution $\xi^F \subset TF$. Since we assume that all contact structures are co-oriented, there is a 1-form $\alpha^F \in \Omega^1(F)$ with $\ker \alpha^F = \xi^F$ which gives ξ^F its co-orientation. Denote the group of co-orientation preserving contactomorphisms of (F, ξ^F) by $\text{Diff}_+(F, \xi^F)$. That is:

$$\text{Diff}_+(F, \xi^F) = \{\varphi : F \rightarrow F | \varphi^* \alpha^F = e^h \alpha^F \text{ for some } h \in C^\infty(F)\}.$$

Definition 2.1 (provisional). Let (F, ξ^F) be a contact manifold. A fiber bundle $F \rightarrow M \xrightarrow{\pi} B$ is *contact* if the transition maps take values in $\text{Diff}_+(F, \xi^F)$. That is, suppose $\{U_i\}$ is a cover of B by sufficiently small sets. Then there exist trivializations $\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times F$ such

that for all indices i, j with $U_i \cap U_j \neq \emptyset$ and for every $b \in U_i \cap U_j$ the diffeomorphisms $\phi_j \circ \phi_i^{-1}|_{\{b\} \times F} : F \rightarrow F$ are elements of $\text{Diff}_+(F, \xi^F)$.

Remark 2.2. It follows from the definition that for every point $b \in B$ the fiber $F_b := \pi^{-1}(b)$ has a well-defined (co-oriented) contact structure ξ^b . Namely let U_i be an element of the cover containing b . Let $pr_2 : U_i \times F \rightarrow F$ denote the projection on the second factor. Then $\alpha_i := (pr_2 \circ \phi_i)^* \alpha^F \in \Omega(\pi^{-1}(U_i))$ is a 1-form with the restriction $\alpha_i|_{F_b}$ to the fiber being contact. Moreover, if b is also in U_j then $\ker(\alpha_i|_{F_b}) = \ker(\alpha_j|_{F_b})$, so the fiber F_b has a well-defined contact structure ξ^b . We let

$$\xi^\nu = \bigcup_{b \in B} \xi^b \subset \mathcal{V}, \tag{2.1}$$

where ξ^ν is a codimension one subbundle of the vertical bundle $\mathcal{V} := \ker(d\pi : TM \rightarrow TB)$.

Lemma 2.3. Let $F \rightarrow M \xrightarrow{\pi} B$ be a contact fiber bundle as in Definition 2.1. There exists a 1-form α on M such that $(\ker \alpha) \cap \mathcal{V} = \xi^\nu$, where $\mathcal{V} \subset TM$ is the vertical bundle and ξ^ν is the bundle defined by (2.1). In other words, a contact fiber bundle has a globally defined 1-form that restricts to a contact form on each fiber.

Proof. Let $\{U_i\}$ be a sufficiently small open cover of B . Choose a partition of unity ρ_i subordinate to $\{U_i\}$. Let $\alpha_i \in \Omega^1(\pi^{-1}(U_i))$ denote the 1-forms constructed in Remark 2.2. Then $\alpha = \sum_i (\pi^* \rho_i) \alpha_i$ is a globally defined 1-form on M with $(\ker \alpha) \cap T(F_b)$ a contact structure on each fiber F_b . □

Lemma 2.4 and Corollary 2.6 form a converse to Lemma 2.3. Recall that a connection on a fiber bundle $\pi : M \rightarrow B$ is a choice of a complement \mathcal{H} in TM to the vertical bundle \mathcal{V} of π so that $TM = \mathcal{H} \oplus \mathcal{V}$.

Lemma 2.4. Let $F \rightarrow M \xrightarrow{\pi} B$ be a fiber bundle with a co-oriented codimension 1 distribution $\xi \subset TM$ such that for each fiber F_b , the intersection $\xi \cap T(F_b)$ is a contact distribution on F_b . Then

1. there is a natural connection $\mathcal{H} = \mathcal{H}(\xi)$ on the fiber bundle $F \rightarrow M \xrightarrow{\pi} B$;
2. the parallel transport with respect to \mathcal{H} (when it exists) preserves the contact structure on the fibers. Additionally parallel transport is co-orientation preserving.

Proof. Since ξ is co-oriented there is a 1-form α on M with $\ker \alpha = \xi$. Let $\omega = d\alpha|_\xi$. Since fiber restrictions $\alpha|_{F_b}$ are contact, $\omega|_{\xi^\nu}$ is nondegenerate, where ξ^ν is the intersection of the distribution ξ with the vertical bundle \mathcal{V} . We define \mathcal{H} to be the ω -perpendicular to ξ^ν in (ξ, ω) :

$$\mathcal{H} = (\xi^\nu)^\omega.$$

Note that if α' is another 1-form with $\ker \alpha' = \xi$ giving ξ its co-orientation then $\alpha' = e^f \alpha$ for some function $f \in C^\infty(M)$. Hence $d\alpha'|_\xi = e^f (d\alpha|_\xi)$ and consequently the definition

of \mathcal{H} does not depend on the choice of α . Since $\omega|_{\xi^\nu}$ is nondegenerate:

$$\xi = \xi^\nu \oplus \mathcal{H} \tag{2.2}$$

and since $\xi^\nu = \xi \cap \mathcal{V}$, \mathcal{H} is a connection on $\pi : M \rightarrow B$.

We now argue that parallel transport defined by \mathcal{H} preserves ξ^ν . (Here we tacitly assume that the parallel transport exists, i.e. that the connection \mathcal{H} is complete. If the fiber F is compact then \mathcal{H} is complete, but this need not be true in full generality. If the parallel transport does not exist globally, it does exist locally: one can parallel transport for short times small pieces of the fibers. The statement of the lemma then becomes messy. And so we gloss over this point here and elsewhere in the paper.) Let v be a vector field on B , let $v^\#$ denote its horizontal lift to M with respect to \mathcal{H} : for each $m \in M$, $v^\#(m)$ is the unique vector field in $\mathcal{H}_m \subset T_m M$ with $d\pi(v^\#(m)) = v(\pi(m))$. Let w be a section of ξ^ν . We will argue that the Lie bracket $[v^\#, w]$ is also a section of ξ^ν . Since $v^\#$ is a horizontal lift and w is vertical, the bracket $[v^\#, w]$ is also vertical: $[v^\#, w] \in \Gamma(\mathcal{V})$. By definition of \mathcal{H} , $0 = \omega(v^\#, w) = d\alpha(v^\#, w)$. Since $v^\#, w \in \Gamma(\xi)$, we have $\iota(w)\alpha = 0 = \iota(v^\#)\alpha$. Therefore, $0 = d\alpha(v^\#, w) = v^\#(\alpha(w)) - w(\alpha(v^\#)) - \alpha([v^\#, w])$. Hence $\alpha([v^\#, w]) = 0$, i.e. $[v^\#, w] \in \xi$. Consequently, $[v^\#, w] \in \Gamma(\mathcal{V}) \cap \Gamma(\xi) = \Gamma(\mathcal{V} \cap \xi) = \Gamma(\xi^\nu)$, and so the parallel transport with respect to \mathcal{H} preserves ξ^ν .

Finally we argue that the parallel transport also preserves the co-orientation. This is a continuity argument. Let $\gamma : [0, 1] \rightarrow B$ be a path, $\mathcal{P}_{\gamma(t)} : F_{\gamma(0)} \rightarrow F_{\gamma(t)}$ be the parallel transport along γ . Since $d\mathcal{P}_{\gamma(t)}(\ker(\alpha|_{F_{\gamma(0)}})) = \ker(\alpha|_{F_{\gamma(t)}})$, we have

$$(\mathcal{P}_{\gamma(t)})^*(\alpha|_{F_{\gamma(t)}}) = f_t(\alpha|_{F_{\gamma(0)}})$$

for some nowhere zero function $f_t \in C^\infty(F_{\gamma(0)})$ depending continuously on t . Since $f_0 = 1 > 0$, $f_t > 0$ for all $t \in [0, 1]$. □

Definition 2.5. We will refer to the connection $\mathcal{H} = \mathcal{H}(\xi)$ defined in Lemma 2.4 as a *contact connection*.

Corollary 2.6. Let $F \rightarrow M \xrightarrow{\pi} B$ be a fiber bundle with a co-oriented codimension 1 distribution $\xi \subset TM$ such that for each fiber F_b , the intersection $\xi \cap T(F_b)$ is a contact distribution on F_b and the associated contact connection is complete. Then $F \rightarrow M \xrightarrow{\pi} B$ is a contact fiber bundle in the sense of Definition 2.1.

Proof. Use parallel transport with respect to the contact connection \mathcal{H} to define local trivializations of the fiber bundle $\pi : M \rightarrow B$. □

Remark 2.7. Corollary 2.6 shows that we may also define a contact fiber bundle to be a fiber bundle $F \rightarrow M \xrightarrow{\pi} B$ with a co-oriented codimension 1 distribution $\xi \subset TM$ such that $\xi \cap T(F_b)$ is a contact structure on each fiber F_b of π . From now on we will take this as our definition of a contact fiber bundle as opposed to Definition 2.1.

We finish this section by pointing out that contact fiber bundles as defined in the remark above are easily constructible as associated bundles.

Lemma 2.8. *Let (F, ξ^F) be a contact manifold with an action of a Lie group G preserving the contact structure ξ^F and its co-orientation. Let $G \rightarrow P \rightarrow B$ be a principal G -bundle. For any (G -invariant) connection $\mathcal{H} \subset TP$ the distribution:*

$$\xi_M := \tilde{\mathcal{H}} \oplus (P \times_G \xi^F) \tag{2.3}$$

makes $M = P \times_G F$ into a contact fiber bundle. Here $\tilde{\mathcal{H}} \subset TM$ is the connection on $M \rightarrow B$ induced by \mathcal{H} .

Proof. See proof of Theorem 3.6. □

3. When is a contact fiber bundle a contact manifold?

We now consider the conditions on a contact fiber bundle $(F \rightarrow M \xrightarrow{\pi} B, \xi \subset TM)$ that ensure that ξ is a contact structure on the total space M . We will see that the question is related to the fatness of the contact connection $\mathcal{H}(\xi)$ and the image of the moment map for the action of the structure group of $\pi : M \rightarrow B$ on the fiber F .

Recall that the curvature of a connection \mathcal{H} on a fiber bundle $F \rightarrow M \xrightarrow{\pi} B$ is a 2-form $\text{Curv}_{\mathcal{H}}$ on B with values in the vector fields on the fiber: for $x, y \in T_b B$ and vector fields $v, w \in \chi(B)$ with $v(b) = x, w(b) = y$

$$(\text{Curv}_{\mathcal{H}})_b(x, y) := [v^{\#}, w^{\#}] - [v, w]^{\#} \in \chi(F_b),$$

where, as in Lemma 2.4, $\#$ denotes the horizontal lift with respect to \mathcal{H} .

Proposition 3.1. *Let $(F \rightarrow M \xrightarrow{\pi} B, \xi \subset TM)$ be a contact fiber bundle, $\mathcal{H} = \mathcal{H}(\xi)$ the contact connection, and $\xi^{\circ} \subset T^*M$ the annihilator of ξ . The distribution ξ is a contact structure on M iff for all $m \in M$ and all $0 \neq \eta \in \xi_m^{\circ}$:*

$$\langle \eta, [(\text{Curv}_{\mathcal{H}})_b(\cdot, \cdot)](m) \rangle : T_b B \times T_b B \rightarrow \mathbb{R} \text{ is nondegenerate,} \tag{3.1}$$

where $b = \pi(m)$.

Proof. Choose a 1-form α on M with $\ker \alpha = \xi$. The distribution ξ is contact iff $d\alpha|_{\xi}$ is nondegenerate. Now $\xi = \mathcal{H} \oplus \xi^v$ (cf. (2.2)) and $d\alpha|_{\xi^v}$ is nondegenerate since $(F \rightarrow M \xrightarrow{\pi} B, \xi)$ is a contact fiber bundle. Consequently, ξ is contact iff $d\alpha|_{\mathcal{H}}$ is nondegenerate. Now fix a point $m \in M$ and two vectors $x, y \in \mathcal{H}_m \subset T_m M$. Choose vector fields v, w on B with $v^{\#}(m) = x, w^{\#}(m) = y$ ($\#$ denotes the horizontal lift). Then, omitting evaluations at m , we have

$$\begin{aligned} d\alpha(x, y) &= d\alpha(v^{\#}, w^{\#}) = v^{\#}(\alpha(w^{\#})) - w^{\#}(\alpha(v^{\#})) - \alpha([v^{\#}, w^{\#}]) \\ &= 0 - 0 - \alpha([v^{\#}, w^{\#}] - [v, w]^{\#}), \end{aligned}$$

since $\alpha(u^{\#}) = 0$ for any vector field u on B . Since $(\text{Curv}_{\mathcal{H}})(v, w) = [v^{\#}, w^{\#}] - [v, w]^{\#}$, we have

$$(d\alpha_m|_{\mathcal{H}_m})(x, y) = d\alpha_m(x, y) = \langle \alpha_m, [(\text{Curv}_{\mathcal{H}})_{\pi(m)}(d\pi(x), d\pi(y))] \rangle(m).$$

For any $0 \neq \eta \in \xi_m^\circ$ there is $s \neq 0$ such that $\eta = s\alpha_m$. Therefore, $d\alpha_m|_{\mathcal{H}_m}$ is nondegenerate iff (3.1) holds for any $0 \neq \eta \in \xi_m^\circ$. \square

We now reinterpret (3.1) in terms of contact moment maps and fatness.

3.1. Contact moment maps

Consider a manifold F with a co-oriented contact structure ξ^F . As mentioned in Remark 1.1, the punctured annihilator bundle $(\xi^F)^\circ \setminus F$ has two components: $(\xi^F)^\circ \setminus F = (\xi^F)_+^\circ \sqcup -(\xi^F)_+^\circ$. Consider the Lie algebra of contact vector fields $\chi(F, \xi^F)$ on F . Recall that the contact vector fields are in one-to-one correspondence with sections of the line bundle $TF/\xi \rightarrow M$: the map $\chi(F, \xi) \rightarrow \Gamma(TF/\xi)$, $v \mapsto v + \xi$ gives the bijection. This is a standard fact. See for example [5]. There is a natural pairing between the points of the line bundle $(\xi^F)^\circ \rightarrow F$ and the contact vector fields:

$$(\xi^F)^\circ \times \chi(F, \xi^F) \rightarrow \mathbb{R}, \quad ((f, \eta), v) \mapsto \langle \eta, v(f) \rangle \tag{3.2}$$

for all $f \in F$, $\eta \in ((\xi^F)^\circ)_f$ and $v \in \chi(F, \xi^F)$, where on the right the pairing is between a covector $\eta \in (\xi^F)_f \subset T_f^*F$ and a vector $v(f) \in T_fF$.

Suppose a Lie algebra \mathfrak{g} acts on F by contact vector fields, i.e. suppose there is a representation $\rho : \mathfrak{g} \rightarrow \chi(F, \xi)$ (or an anti-representation; this is the usual problem with left actions and Lie brackets defined in terms of left invariant vector fields). Then the moment map $\Psi : \chi(F, \xi)^* \supset (\xi^F)^\circ \rightarrow \mathfrak{g}^*$ should be the transpose of ρ relative to the pairing (3.2) and the natural pairing $\mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$. However, for various reasons (see below) in the case of co-oriented contact structures, it is more convenient to define the contact moment map for ρ as a map $\Psi : (\xi^F)_+^\circ \rightarrow \mathfrak{g}^*$:

$$\langle \Psi(f, \eta), X \rangle = \langle (f, \eta), \rho(X) \rangle = \langle \eta, \rho(X)(f) \rangle. \tag{3.3}$$

Note that $(\xi^F)_+^\circ$ is a symplectic submanifold of the cotangent bundle T^*F . Suppose the anti-representation $\rho : \mathfrak{g} \rightarrow \chi(F, \xi)$ comes from a (left) action of a Lie group G on F preserving ξ and its co-orientation (with Lie algebra of G being \mathfrak{g}). In this case we write X_F for $\rho(X)$. The moment map $\Psi : (\xi^F)_+^\circ \rightarrow \mathfrak{g}^*$ is simply the restriction to $(\xi^F)_+^\circ$ of the moment map $\Phi : T^*F \rightarrow \mathfrak{g}^*$ for the lifted action of G on T^*F . The action of G preserves ξ^F and its co-orientation if and only if the lifted action preserves $(\xi^F)_+^\circ$. A G -invariant 1-form α^F on F with $\ker \alpha^F = \xi^F$ is a G -equivariant section of the bundle $(\xi^F)_+^\circ \rightarrow F$. Therefore, the composition

$$\Psi_{\alpha^F} := \Psi \circ \alpha^F : F \rightarrow \mathfrak{g}^* \tag{3.4}$$

is a G -equivariant map. We will refer to it as the α^F -moment map. Note that by definition

$$\langle \Psi_{\alpha^F}, X \rangle(f) = \langle \alpha^F_f, X_F(f) \rangle = \langle \iota(X_F)\alpha^F, f \rangle \tag{3.5}$$

for all $X \in \mathfrak{g}$ and all $f \in F$. This is the “classical” definition of a contact moment map (cf. [1–3]). It depends on a choice of a contact form, unlike $\Psi : (\xi^F)_+^\circ \rightarrow \mathfrak{g}^*$, which is “universal.”

Remark 3.2. The pairing (3.2) suggests a different way of looking at Proposition 3.1. Denote by ξ^b the contact structure on the fiber $F_b : \xi^b = \xi \cap T(F_b)$. Then for a point $m \in F_b$, a covector $\eta \in (\xi^b)_m^\circ$ and a vector field $v \in \chi(F_b, \xi^b)$ we have

$$\langle (m, \eta), v \rangle = \langle \eta, v(m) \rangle \in \mathbb{R}.$$

Note that the connection \mathcal{H} allows us to identify ξ° with $\bigcup_b (\xi^b)^\circ$ and consequently $\xi_+^\circ = \bigcup_b (\xi^b)_+^\circ$. Consequently, the curvature $\text{Curv}_{\mathcal{H}}$ of the connection \mathcal{H} gives rise to a well-defined skew-symmetric form $\sigma_{\mathcal{H}}$ on the vector bundle $\tilde{\mathcal{H}} \rightarrow \xi_+^\circ$ which is the pull-back of $\mathcal{H} \rightarrow M$ by the projection $p : \xi_+^\circ \rightarrow M$. Namely, since $\mathcal{H} \simeq \pi^*(TB)$, $\tilde{\mathcal{H}} = (\pi \circ p)^*(TB)$. So given $m \in M$, $\eta \in (\xi_+^\circ)_m$ and $u, v \in T_b B$, where $b = \pi(m)$,

$$(\sigma_{\mathcal{H}})_{(m,\eta)}(u, v) := \langle (m, \eta), (\text{Curv}_{\mathcal{H}})_b(u, v) \rangle = \langle \eta, [(\text{Curv}_{\mathcal{H}})_b(u, v)](m) \rangle. \tag{3.6}$$

Thus Proposition 3.1 asserts:

The distribution ξ is a contact structure on M if and only if $\sigma_{\mathcal{H}}$ defined above is a symplectic form on the vector bundle $\tilde{\mathcal{H}} \rightarrow \xi_+^\circ$.

$$\tag{3.7}$$

3.2. Fatness

Definition 3.3 (Weinstein [6]). A connection 1-form A on a principal G -bundle $G \rightarrow P \xrightarrow{\pi} B$ is fat at a point $\mu \in \mathfrak{g}^*$ if for any point $p \in P$ the bilinear map $\mu \circ (\text{Curv}_A)_p : \mathcal{H}_p^A \times \mathcal{H}_p^A \rightarrow \mathbb{R}$ is nondegenerate, where Curv_A is the curvature of the connection form A and $\mathcal{H}_p^A = \ker(A_p : T_p P \rightarrow \mathfrak{g})$ is the associated horizontal distribution.

Remark 3.4. If A is fat at $\mu \in \mathfrak{g}^*$, it is fat at every point of the set $\{\text{Ad}^\dagger(g)(a\mu) | g \in G, a > 0\}$ (here and elsewhere Ad^\dagger denotes the coadjoint action).

Remark 3.5. Fatness is an open condition. Thus if A is fat at μ , it is fat at every point of a $G \times \mathbb{R}^+$ -invariant neighborhood of μ in \mathfrak{g}^* .

Theorem 3.6. Suppose a Lie group G acts [on the left] on a manifold F preserving a contact distribution ξ^F and its co-orientation; let $\Psi : (\xi^F)_+^\circ \rightarrow \mathfrak{g}^*$ denote the associated moment map. Let $G \rightarrow P \rightarrow B$ be a principal G -bundle. Given a connection 1-form A on P , there exists a co-oriented codimension 1 distribution ξ on the associated bundle $M = P \times_G F \rightarrow B$ which intersects the tangent bundle of each fiber F_b in a contact distribution isomorphic to ξ^F . Explicitly

$$\xi = \mathcal{H} \oplus (P \times_G \xi^F), \tag{3.8}$$

where \mathcal{H} is the connection on $\pi : M \rightarrow B$ induced by A .

Moreover, the distribution ξ is a contact structure on M if and only if the connection A is fat at the points of the image $\Psi((\xi^F)_+^\circ)$.

Proof. Since the action of G on F preserves ξ^F and its co-orientation

$$\xi^\mathcal{V} := P \times_G \xi^F$$

is a well-defined co-oriented subbundle of the vertical bundle $\mathcal{V} \simeq P \times_G (TF)$ of $M \rightarrow B$. The connection 1-form A defines a complement \mathcal{H} to \mathcal{V} in TM . Therefore, the distribution ξ on M defined by (3.8) is a co-oriented codimension 1 distribution. By construction, for each fiber F_b we have $\xi \cap T(F_b) = \xi^\mathcal{V}|_{F_b} \simeq \xi^F$. (More precisely, for each point $p \in P$ we have an embedding $\iota_p : F \hookrightarrow M$, $\iota_p(f) = [p, f]$ (where $[p, f] \in P \times_G F$ denotes the image of $(p, f) \in P \times F$). Then $d\iota_p(\xi^F) = \xi^\mathcal{V}|_{F_b}$ where $b = \pi(p)$.)

Now suppose that A is fat at the points of $\Psi((\xi^F)_+^\circ)$. By Proposition 3.1 and Remark 3.2 it enough to show that for any $[p, f, \eta] \in P \times_G (\xi^F)_+^\circ = \xi_+^\circ$, the pairing

$$(\sigma_{\mathcal{H}})_{([p, f, \eta])} = \langle [p, f, \eta], ((\text{Curv}_{\mathcal{H}})_b(\cdot, \cdot))([p, f]) \rangle : T_b B \times T_b B \rightarrow \mathbb{R}$$

is nondegenerate (where $b = \pi([p, f])$).

The curvature $\text{Curv}_A : \mathcal{H}^A \times \mathcal{H}^A \rightarrow \mathfrak{g}$ of A defines a 2-form $\overline{\text{Curv}}_A$ on B with values in the adjoint bundle $P \times_G \mathfrak{g}$. To write out $\overline{\text{Curv}}_A \in \Omega^2(B, P \times_G \mathfrak{g})$ explicitly we need a bit of notation. For a point $b \in B$ and vectors $x, y \in T_b B$ denote the horizontal lift of x and y to \mathcal{H}_p^A by $x_p^\#$ and $y_p^\#$, respectively. Then

$$(\overline{\text{Curv}}_A)_b(x, y) = [p, (\text{Curv}_A)_p(x_p^\#, y_p^\#)] \in P \times_G \mathfrak{g}$$

for any $p \in P$ in the fiber of $P \rightarrow B$ above b .

Since G acts on F by contact transformations, there is an (anti-)representation $\rho : \mathfrak{g} \rightarrow \chi(F, \xi^F)$, $\rho(X) = X_F$, from the Lie algebra \mathfrak{g} of G to contact vector fields on F . Recall: the moment map $\Psi : (\xi^F)_+^\circ \rightarrow \mathfrak{g}$ is the adjoint of ρ in the sense that

$$\langle (f, \eta), \rho(X) \rangle = \langle \Psi(f, \eta), X \rangle$$

for all $f \in F$, $\eta \in (\xi^F)_+^\circ$, $X \in \mathfrak{g}$, where on the right the pairing is the natural pairing $\mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ and on the left it is the pairing (3.2).

Since ρ and Ψ are equivariant, they induce maps of associated bundles

$$\bar{\Psi} : P \times_G (\xi^F)_+^\circ \rightarrow P \times_G \mathfrak{g}^*, \quad \bar{\Psi}([p, f, \eta]) = [p, \Psi(f, \eta)]$$

and

$$\bar{\rho} : P \times_G \mathfrak{g} \rightarrow P \times_G (\chi(F, \xi^F)), \quad \bar{\rho}([p, X]) = [p, \rho(X)].$$

The two pairings above give rise to fiber-wise pairings:

$$(P \times_G \mathfrak{g}^*) \oplus (P \times_G \mathfrak{g}) \rightarrow P \times_G \mathbb{R} = B \times \mathbb{R}, \quad [p, \mu] \oplus [p, X] \mapsto (\pi(p), \langle \mu, X \rangle)$$

and

$$P \times_G ((\xi^F)_+^\circ \times \chi(F, \xi^F)) \rightarrow P \times_G \mathbb{R} = B \times \mathbb{R}, \\ [p, ((f, \eta), v)] \mapsto (\pi(p), \langle \eta, v(f) \rangle),$$

where π now denotes the projection $P \rightarrow B$. The maps $\bar{\rho}$ and $\bar{\Psi}$ are adjoint with respect to the two fiber-wise pairings.

Finally, the map $\bar{\rho}$ relates the two curvatures, $\overline{\text{Curv}}_A \in \Omega^2(P \times_G \mathfrak{g})$ and $\text{Curv}_{\mathcal{H}} \in \Omega^2(B, P \times_G \chi(F, \xi^F))$:

$$\bar{\rho} \circ \overline{\text{Curv}}_A = \text{Curv}_{\mathcal{H}}.$$

Putting together the above remarks we get

$$\begin{aligned} (\sigma_{\mathcal{H}})_{[p, f, \eta]}(u, v) &= \langle [p, f, \eta], (\text{Curv}_{\mathcal{H}})_b(u, v) \rangle \\ &= \langle [p, f, \eta], \bar{\rho} \circ (\overline{\text{Curv}}_A)_b(u, v) \rangle \\ &= \langle \bar{\Psi}([p, f, \eta]), (\overline{\text{Curv}}_A)_b(u, v) \rangle \\ &= \langle \Psi(f, \eta), (\text{Curv}_A)_p(u_p^\#, v_p^\#) \rangle \end{aligned}$$

for all $[p, f, \eta] \in P \times_G (\xi^F)_+^\circ$ and any $u, v \in T_b B$ ($b = \pi([p, f]) \in B$). Thus A is fat at the points of $\Psi((\xi^F)_+^\circ)$ if and only if $\sigma_{\mathcal{H}}$ is nondegenerate. \square

Remark 3.7. Theorem 3.6 allows us to reinterpret (3.7). Namely, suppose $(F \rightarrow M \rightarrow B, \xi)$ is a contact fiber bundle and $\mathcal{H}(\xi)$ is the corresponding contact connection. Suppose the holonomy group G of $\mathcal{H}(\xi)$ is a finite-dimensional Lie group. Then M is an associated bundle for a principal G -bundle $G \rightarrow P \rightarrow B$ and $\mathcal{H}(\xi)$ is induced by a connection A on P . Also, the action of G on a typical fiber (F, ξ^F) is contact and co-orientation preserving. Then by Theorem 3.6, the distribution ξ is a contact structure if and only if A is fat at the points of the image of the moment map $\Psi : (\xi^F)_+^\circ \rightarrow \mathfrak{g}^*$ associated to the action of the holonomy group on a typical fiber.

In general this gives us a formal interpretation of (3.7) as fatness of the connection on the principal G -bundle $P \rightarrow B$, where G is the group of co-orientation preserving contactomorphisms $\text{Diff}_+(F, \xi^F)$ and P is the “frame bundle” of the fiber bundle $M \rightarrow B$.

Remark 3.8. Suppose F is a manifold with an action of a Lie group G and α_F is a G -invariant 1-form on F . Then a choice of a connection 1-form A on a principal G -bundle $G \rightarrow P \rightarrow B$ defines a 1-form $\alpha_M = \alpha_M(A, \alpha_F)$ on the associated bundle $M := P \times_G F$ such that α_M restricted to each fiber of $M \rightarrow B$ is α_F :

Define a moment map $\Psi_{\alpha_F} : F \rightarrow \mathfrak{g}^*$ by

$$\langle \Psi_{\alpha_F}(f), X \rangle = \alpha_F(X_F)$$

for all $X \in \mathfrak{g}$, where X_F denotes the vector field induced by X on F (cf. (3.5)). It is easy to check that the 1-form α on $P \times F$ given by

$$\alpha_{(p, f)} = \langle \Psi_{\alpha_F}(f), A_p \rangle + (\alpha_F)_f$$

is basic relative to the diagonal action of G on $P \times F$:

$$g \cdot (p, f) = (p \cdot g^{-1}, g \cdot f).$$

(Since P is a principal G -bundle, the natural action of G on P is a right action: $(g, p) \mapsto p \cdot g$. The diagonal action of G above is a left action. This matters because of the signs below.) Now, for any $X \in \mathfrak{g}$,

$$\iota(X_{P \times F})\alpha = \langle \Psi_{\alpha_F}, A(X_P) \rangle + \iota(X_F)\alpha_F = \langle \Psi_{\alpha_F}, -X \rangle + \langle \Psi_{\alpha_F}, X \rangle = 0.$$

Since α is G -invariant, it descends to 1-form α_M on M .

By **Theorem 3.6**, if α_F is a contact form and if the connection A is fat at the points of $\Psi_{\alpha_F}(F)$, then α_M is a contact form. Moreover, in this case $\xi = \ker \alpha_M$ is precisely the distribution on M defined by **Eq. (2.3)**.

In the rest of the paper we discuss two applications of **Theorem 3.6**— K -contact fiber bundles and contact cross-sections.

4. Application 1: K -contact fiber bundles

Definition 4.1. Let (F, ξ) be a contact co-oriented manifold. It is K -contact if there is a metric g on F such that

1. the unit normal n to the contact distribution ξ , which is defined by the metric g and the co-orientation $\xi_+^\circ \subset \xi^\circ$ of ξ , is Killing, i.e. $L_n g = 0$;
2. the contact form α_g given by $\alpha_g = g(n, \cdot)$ is compatible with g in the sense that

$$d\alpha_g|_\xi = (g|_\xi)(\cdot, J\cdot) \tag{4.1}$$

for some complex structure J on ξ with $J^*(g|_\xi) = g|_\xi$.

We will refer to the triple (F, ξ_+°, g) as a K -contact structure and to g as a K -contact metric on (F, ξ) .

Note that the vector field n in the definition above is the Reeb vector field of the contact form α_g .

Remark 4.2. Given a contact form α on a manifold F , we can easily find a metric g on F such that the Reeb vector field R_α of α is unit and normal to $\xi = \ker \alpha$ and such that α and g are compatible ((4.1) holds). If R_α happens to be Killing with respect to g then g is a K -contact metric.

We now relate, following Yamazaki, K -contact structures on compact manifolds and contact torus actions (cf. [9, Proposition 2.1]).

Proposition 4.3. A compact contact co-orientable manifold (F, ξ) admits a K -contact metric g if and only if there is an action of a torus T on F preserving ξ and a vector X in the Lie algebra \mathfrak{t} of T so that the function $\langle \Psi, X \rangle : \xi_+^\circ \rightarrow \mathbb{R}$ is strictly positive. Here $\Psi : \xi_+^\circ \rightarrow \mathfrak{t}^*$ is the moment map associated with the action of T on (F, ξ) .

Proof. Suppose there is an action of a torus T on (F, ξ) and $X \in \mathfrak{t}$ such that $\langle \Psi, X \rangle : \xi_+^\circ \rightarrow \mathbb{R}$ is strictly positive. Since the action of T preserves ξ , the lifted action of T on T^*F preserves ξ° . Since T is connected, the lifted action preserves a component ξ_+° of $\xi \setminus F$. It follows that for any 1-form β on F with $\ker \beta = \xi$, the average $\bar{\beta}$ of β over T still satisfies $\ker \beta = \xi$ (if the action of a group does not preserve the co-orientation of ξ , the average of β may be zero at some points). Hence we may assume that there is a T -invariant 1-form α' with

$\alpha'(F) \subset \xi_+^\circ$. Now let

$$\alpha = (\langle \Psi \circ \alpha', X \rangle)^{-1} \alpha'.$$

Then, since $\iota(X_F)\alpha' = \langle \Psi \circ \alpha', X \rangle$ (cf. Eqs. (3.4) and (3.5)), $\iota(X_F)\alpha = 1$. Then $TF = \xi \oplus \mathbb{R}X_F$ and the splitting is T -equivariant. We use the splitting to define the desired metric g . We declare ξ and $\mathbb{R}X_F$ to be orthogonal and define $g(X_F, X_F) = 1$, so that X_F is a unit normal to ξ . On ξ we choose a T -invariant complex structure J compatible with $d\alpha|_\xi$ and define $g|_{\xi}(\cdot, \cdot) = d\alpha|_{\xi}(\cdot, J\cdot)$. Then g is T -invariant and hence $L_{X_F}g = 0$. Thus g is a K -contact metric on (F, ξ) .

Conversely, if there is a metric g on F making (F, ξ) K -contact, the flow $\{\exp(tn)\}$ of the unit normal vector field n to ξ is a 1-parameter group of isometries $\text{Diff}(F, g)$. Since F is compact $\text{Diff}(F, g)$ is a compact Lie group. Hence the closure $T = \overline{\{\exp(tn)\}}$ is a compact connected abelian Lie group, i.e. a torus. Let X be the vector in the Lie algebra \mathfrak{t} of T with $X_F = n$. Let $\alpha = g(n, \cdot)$. Then $\langle \Psi \circ \alpha, X \rangle = \iota(X_F)\alpha = g(n, n) = 1$. Hence $\langle \Psi, X \rangle > 0$. □

Theorem 4.4. *Let $(F, (\xi^F)_+^\circ, g_F)$ be a compact K -contact manifold. Let $G \subset \text{Diff}(F, g_F)$ be a group of isometries preserving $(\xi^F)_+^\circ$. Let $\Psi : (\xi^F)_+^\circ \rightarrow \mathfrak{g}^*$ denote the associated moment map. Suppose a principal G -bundle $P \rightarrow B$ has a connection 1-form A which is fat at the points of the image $\Psi((\xi^F)_+^\circ)$. Then there exists a K -contact structure on the associated bundle $M = P \times_G F$ compatible with the contact form $\alpha_M = \alpha_M(A, \alpha_{g_F})$ (the form constructed in Remark 3.8). Here α_{g_F} is the contact form on F defined by g_F and $(\xi^F)_+^\circ$.*

Proof. As we saw above the isometry group $\text{Diff}(F, g_F)$ is a compact Lie group. Also the flow $\{\exp(tn)\}$ of the unit normal n is a subgroup of the isometry group whose closure $T = \overline{\{\exp(tn)\}}$ is a torus. Since the normal n is G -invariant, T and G commute inside $\text{Diff}(F, g_F)$. Therefore, the torus T acts naturally on $M = P \times_G F : a \cdot [p, f] = [p, a \cdot f]$ for all $(p, f) \in P \times F$ and all $a \in T$.

The Reeb vector field R of α_M is tangent to fibers of $M \rightarrow B$, hence $R|_{F_b}$ is the Reeb vector field of $\alpha_M|_{F_b}$ for any fiber F_b . Consequently, R is the vector field induced on M by the G -invariant vector field $n \in \chi(F)^G$. Hence the flow of R generates the action of T on M . Therefore, the K -contact metric g on M has to be T -invariant. Conversely, any T -invariant metric g on M compatible with α_M is K -contact. The action of T on M preserves the horizontal subbundle $\mathcal{H} \subset TM$ defined by A , and it preserves the symplectic structure $d\alpha_M|_{\mathcal{H}}$. Choose a T -invariant complex structure $J_{\mathcal{H}}$ on \mathcal{H} compatible with $d\alpha_M|_{\mathcal{H}}$. The $g_{\mathcal{H}} := d\alpha_M|_{\mathcal{H}}(J_{\mathcal{H}}\cdot, \cdot)$ is a T -invariant metric on \mathcal{H} . The $T \times G$ -invariant metric g_F on F gives rise to a T -invariant metric $g_{\mathcal{V}}$ on the vertical bundle \mathcal{V} of $M \rightarrow B$. The metric $g_M := g_{\mathcal{H}} \oplus g_{\mathcal{V}}$ is a T -invariant metric on M compatible with α_M (recall that $\xi := \ker \alpha_M = \mathcal{H} \oplus (P \times_G \xi^F)$). Moreover, the Reeb vector field R of α_M is unit, normal to ξ and Killing with respect to g_M . Thus (M, ξ_+°, g_M) is a K -contact structure on $F \rightarrow M = P \times_G F \rightarrow B$. □

Remark 4.5. There is no natural way to make the map $P \times_T F \rightarrow B$ into a Riemannian submersion relative to the K -contact metric on $P \times_T F$ produced by Theorem 4.4. Indeed,

if we trace through the construction of g_M we will see that for any point $[p, f]$ in the fiber F_b of $P \times_G F \rightarrow B$ we have

$$(g\mathcal{H})_{[p,f]}(v^\#, u^\#) = \langle \Psi_{\alpha_{gF}}(f), dA_p(v^\#, J_{\mathcal{H}}u^\#) \rangle \tag{4.2}$$

for any tangent vectors $u, v \in T_bB$. Here on the left-hand side $v^\#, u^\#$ denote horizontal lifts to $\mathcal{H}_{[p,f]}$. On the right-hand side $v^\#, u^\#$ denote horizontal lifts to $\ker A_p \subset T_pP$. Thus the horizontal part of the metric g_M depends on the points in the fiber F_b !

Example 4.6. Let Σ be a compact Riemann surface and $\omega \in \Omega^2(\Sigma)$ an area form which is integral, i.e. $\int_\Sigma \omega \in \mathbb{Z}$. Let $S^1 \rightarrow P \xrightarrow{\pi} \Sigma$ be the corresponding principal circle bundle with a connection 1-form $A \in \Omega(P, \mathbb{R})^{S^1}$ satisfying $dA = \pi^*\omega$. Then A is fat on $\mathbb{R} \setminus \{0\}$. Let α be a contact form on a manifold F such that the flow of the Reeb vector field R_α is periodic. For example, we may take F to be the odd-dimensional sphere $S^{2n-1} = \{z \in \mathbb{C}^n \mid \|z\|^2 = 1\}$ with the standard contact form $\alpha = \sqrt{-1}(\sum z_j d\bar{z}_j - \bar{z}_j dz_j)|_{S^{2n-1}}$. Or we can take F to be the co-sphere bundle $S(T^*S^k)$ of a sphere with the contact form defined by the standard round metric on S^k . Then the associated bundle $P \times_{S^1} F$ is a K -contact manifold.

The next example is a slight generalization. It produces K -contact manifolds first constructed by Yamazaki by a “fiber join” [7].

Example 4.7. For an n -tuple $a = (a_1, a_2, \dots, a_n)$, $a_j > 0$, the ellipsoid $E_a := \{z \in \mathbb{C}^n \mid \sum a_j |z_j|^2 = 1\} \simeq S^{2n-1}$ is a hypersurface of contact type in \mathbb{C}^n . The corresponding contact form α_a is given by $\alpha_a := \sqrt{-1}(\sum z_j d\bar{z}_j - \bar{z}_j dz_j)|_{E_a}$. For a generic a , the Reeb vector field of α_a generates the action of the n -torus \mathbb{T}^n . The image of E_a under the α_a -moment map is the simplex

$$\Delta_a = \{(t_1, \dots, t_n) \in \mathbb{R}^n \simeq \text{Lie}(\mathbb{T}^n)^* \mid \sum a_j t_j = 1, \quad t_j \geq 0\}.$$

Suppose $\omega_1, \dots, \omega_n$ are integral symplectic forms on a compact Riemann surface Σ such that $\sum t_j \omega_j$ is nondegenerate for all $t = (t_1, \dots, t_n) \in \Delta_a$. For example, we may pick one integral area form ω and let $\omega_j = \omega$ for all j . Then the principal \mathbb{T}^n -bundle P over Σ defined by $\omega_1, \dots, \omega_n$ has a connection 1-form $A = (A_1, \dots, A_n) \in \Omega(P, \mathbb{R}^n)^{\mathbb{T}^n}$ with $dA = (\pi^*\omega_1, \dots, \pi^*\omega_n)$. The connection A is fat at the points of Δ_a . Therefore, $P \times_{\mathbb{T}^n} E_a$ has a K -contact structure. It is an S^{2n-1} -bundle over Σ .

5. Application 2: contact cross-sections

Let M be a manifold with an action of a compact connected Lie group G preserving a (co-oriented) contact structure ξ on M . Then there exists a G -invariant 1-form α with $\ker \alpha = \xi$. (Pick any 1-form α' with $\ker \alpha' = \xi$ and average it over G . Since ξ is G -invariant and since G is connected the averaged form α satisfies $\ker \alpha = \xi$.) Denote by $\Psi_\alpha : M \rightarrow \mathfrak{g}^*$ the associated α -moment map: $\langle \Psi_\alpha(x), X \rangle = \alpha_x(X_M(x))$ for all $x \in M$ and all $X \in \mathfrak{g}$; cf. (3.5).

Since G is compact, for any point $\mu \in \mathfrak{g}^*$ the isotropy Lie algebra \mathfrak{g}_μ of μ has a G_μ -invariant complement \mathfrak{m} in \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m} \quad (G_\mu\text{-equivariant}). \tag{5.1}$$

Moreover, we may choose \mathfrak{m} so that $\mu|_{\mathfrak{m}} = 0$, i.e. $\mu \in \mathfrak{m}^\circ$. (Pick a G -invariant metric on \mathfrak{g} and let $\mathfrak{m} = \mathfrak{g}_\mu^\perp$.) Then a large $\mathbb{R}^{>0}$ -invariant open subset \mathcal{S} of \mathfrak{m}° is a slice for the coadjoint action of G at μ . For example, if μ is generic, \mathfrak{g}_μ is a Cartan subalgebra and \mathcal{S} is a Weyl chamber (after some identifications).

We will need (see [6, p. 37] for a proof):

Lemma 5.1. *For any $\eta \in \mathcal{S}$ the pairing*

$$\omega_\eta : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathbb{R}, \quad (X, Y) \mapsto \langle \eta, [X, Y] \rangle.$$

is nondegenerate.

Theorem 5.2. *Let $(M, \xi = \ker \alpha, \Psi_\alpha : M \rightarrow \mathfrak{g}^*)$ be a contact G -manifold, $\mu \in \mathfrak{g}^*$ a point, $\mathfrak{m} \subset \mathfrak{g}$ a subspace defined by (5.1) with $\mu \in \mathfrak{m}^\circ$ and $\mathcal{S} \subset \mathfrak{m}^\circ$ an $\mathbb{R}^{>0}$ -invariant slice for the coadjoint action of G . Define*

$$\mathcal{R} = \Psi_\alpha^{-1}(\mathcal{S}).$$

Then

1. \mathcal{R} is a contact submanifold of (M, ξ) which is independent of the choice of the contact form α used to define it.
2. $G \cdot \mathcal{R}$ is an open subset of M diffeomorphic to the associated bundle $G \times_{G_\mu} \mathcal{R}$.
3. For any $x \in \mathcal{R}$

$$\xi_x = \mathfrak{m}_M(x) \oplus (\xi_x \cap T_x \mathcal{R}). \tag{5.2}$$

In particular, the restriction of the contact structure ξ to $G \cdot \mathcal{R}$ is uniquely determined by the G_μ -invariant contact structure $\xi^{\mathcal{R}} := \xi|_{\mathcal{R}} \cap T\mathcal{R}$.

Remark 5.3. We will refer to the contact submanifold $(\mathcal{R}, \xi^{\mathcal{R}})$ of (M, ξ) as the contact cross-section.

Proof of Theorem 5.2. Since Ψ_α is equivariant, the image $d\Psi_\alpha(T_x M)$ contains the tangent space to the coadjoint orbit $G \cdot \Psi_\alpha(x)$. Since \mathcal{S} is a slice, we have

$$T_\eta \mathfrak{g}^* = T_\eta \mathcal{S} + T_\eta(G \cdot \eta)$$

for any $\eta \in \mathcal{S}$. Hence Ψ_α is transverse to \mathcal{S} , and consequently

$$\mathcal{R} := \Psi_\alpha^{-1}(\mathcal{S})$$

is a submanifold. Also, by equivariance of Ψ_α , \mathcal{R} is G_μ -invariant. Since \mathcal{S} is a slice, $G \cdot \mathcal{S}$ ($\simeq G \times_{G_\mu} \mathcal{S}$) is open in \mathfrak{g}^* . Hence $G \cdot \mathcal{R} = \Psi_\alpha^{-1}(G \cdot \mathcal{S})$ is open in M . Similarly, it is easy to see that $G \cdot \mathcal{R} = G \times_{G_\mu} \mathcal{R}$.

If α' is another G -invariant contact form giving ξ its co-orientation, then $\alpha' = e^f \alpha$ for some function $f \in C^\infty(M)$. Consequently, $\Psi_{\alpha'} = e^f \Psi_\alpha$. Since \mathcal{S} is $\mathbb{R}^{>0}$ -invariant, $\Psi_{\alpha'}^{-1}(\mathcal{S}) = \Psi_\alpha^{-1}(\mathcal{S})$. Thus the cross-section \mathcal{R} does not depend on the choice of the contact form.

Note in passing that $\dim \mathcal{R} = \dim M - \dim G \cdot \mu$, hence odd. In particular \mathcal{R} can be contact.

We next argue that

$$\mathfrak{m}_M(x) := \{X_M(x) | X \in \mathfrak{m}\}$$

is contained in the contact distribution ξ_x for all $x \in \mathcal{R}$. Indeed for any $X \in \mathfrak{m}$

$$\alpha_x(X_M(x)) = \langle \Psi_\alpha(x), X \rangle \in \langle \mathcal{S}, X \rangle \subset \langle \mathfrak{m}^\circ, X \rangle = \{0\},$$

hence $\mathfrak{m}_M(x) \subset \xi_x$.

Fix $x \in \mathcal{R}$. Since $\mathfrak{m}_M(x) \subset \xi_x$ and since $T_x \mathcal{R} \oplus \mathfrak{m}_M(x) = T_x M$, we cannot have $T_x \mathcal{R} \subset \xi_x$. Therefore, $T_x M = T_x \mathcal{R} + \xi$, and consequently

$$\xi_x^{\mathcal{R}} := T_x \mathcal{R} \cap \xi_x$$

is a codimension one subspace of $T_x \mathcal{R}$. The rest of the proof is an argument that $\xi^{\mathcal{R}}$ is indeed a contact structure on \mathcal{R} . In the mean time observe that we have proved (5.2).

We first argue that the restriction $d\alpha|_{\mathfrak{m}_M(x)}$ is nondegenerate for all $x \in \mathcal{R}$. For this we first compute $d\alpha$ on the tangent space of a G -orbit in M . Let $x \in M$ be a point, $X, Y \in \mathfrak{g}$ two vectors, $\eta = \Psi_\alpha(x)$. Then (omitting evaluation at x) we have

$$d\alpha(X_M, Y_M) = X_M(\alpha(Y_M)) - Y_M(\alpha(X_M)) - \alpha([X_M, Y_M]).$$

Now

$$X_M(\alpha(Y_M)) = X_M(\langle \Psi_\alpha, Y \rangle) = \langle d\Psi_\alpha(X_M), Y \rangle.$$

So

$$X_M(\alpha(Y_M))(x) = \langle (d\Psi_\alpha)_x(X_M(x)), Y \rangle = \langle X_{\mathfrak{g}^*}(\eta), Y \rangle = -\langle \eta, [X, Y] \rangle,$$

where the second equality holds by equivariance of the moment map Ψ_α . Similarly,

$$Y_M(\alpha(X_M))(x) = \langle \eta, [X, Y] \rangle.$$

Since $[X_M, Y_M] = -([X, Y])_M$ (we are dealing with a left action!), we have

$$-\alpha([X_M, Y_M])(x) = \langle \eta, [X, Y] \rangle.$$

Putting everything together we get

$$d\alpha(X_M, Y_M)(x) = -\langle \eta, [X, Y] \rangle - \langle \eta, [X, Y] \rangle + \langle \eta, [X, Y] \rangle = -\langle \eta, [X, Y] \rangle.$$

It now follows from Lemma 5.1 that for any $x \in \mathcal{S}$ the restriction $d\alpha|_{\mathfrak{m}_M(x)}$ is nondegenerate.

We next argue that for any $x \in \mathcal{R}$ and any $v \in \xi_x^{\mathcal{R}} = (T_x \mathcal{R} \cap \xi_x)$ and any $X \in \mathfrak{m}$ we have $d\alpha_x(X_M(x), v) = 0$. Let V be a section of $\xi^{\mathcal{R}} \rightarrow \mathcal{R}$ with $V(x) = v$. Then $d\alpha(V, X_M) =$

$V(\alpha(X_M)) - X_M(\alpha(V)) - \alpha([V, X_M])$. Now $\alpha(V) = 0$ and $\alpha([V, X_M]) = 0$ (since ξ is G -invariant $[X_M, V]$ is a section of ξ). Since V is tangent to \mathcal{R} , we have $(d\Psi_\alpha)_x(V(x)) \in T_{\Psi_\alpha(x)}\mathcal{S} = \mathfrak{m}^\circ$ for all $x \in \mathcal{R}$. Therefore, $V(\alpha(X_M))(x) = \langle (d\Psi_\alpha)_x(V(x)), X \rangle \in \langle \mathfrak{m}^\circ, X \rangle = 0$ for all $x \in \mathcal{R}$ since $X \in \mathfrak{m}$. Thus $d\alpha_x(X_M(x), v) = 0$. Consequently

$$\xi_x^{\mathcal{R}} \subset \mathfrak{m}_M(x)^{(d\alpha|_\xi)}.$$

By dimension count the above inclusion is an equality.

Since $d\alpha|_{\mathfrak{m}_M(x)}$ is nondegenerate, $d\alpha|_{\xi_x^{\mathcal{R}}}$ is nondegenerate as well for all $x \in \mathcal{R}$. Thus $\alpha|_{\mathcal{R}}$ is a contact form, $\xi^{\mathcal{R}} = \ker(\alpha|_{\mathcal{R}})$ is a contact structure and \mathcal{R} is a contact submanifold. \square

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